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Ising-Like Field Theory

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A field theory model on R^2 in which the basic fields are Ising spins instead of Gaussian spins is examined. Using statistical mechanics techniques we discuss the ultraviolet and the infrared problems. In particular we discuss a technique yielding the asymptotic expansion in λ of the ground state energy, as $\lambda \to 0$, without using the cluster expansion.

KEY WORDS: Ising model; field theory model; stability; pressure; ultraviolet divergence; infrared divergence.

1. INTRODUCTION AND DESCRIPTION OF THE MODEL

We propose a field theory model which allows us to clarify the statistical mechanics aspects of the ultraviolet and infrared problems that occur in φ^4 Euclidean field theory. In recent years the Markov hierarchical model[(1)] has been introduced with the same purpose. Our model relies on the same basic idea, i.e., to decompose the field in elementary fields of well-defined distribution, but we use Ising model fields at large temperatures instead of Gaussian fields. We will show in two space-time dimensions the stability of the model and the existence of the pressure, corresponding to the ground state density of energy. In the case of the Markov hierarchical model the stability has been shown in two and three dimensions; the existence of the pressure is still an open problem.

We now give a description of the model. Q_N is a partition of R^2 obtained paving the plane with square tesserae of side 2^{-N}; we choose the sequence $\{Q_N\}_{N \geqslant 0}$ such that each tessera $\Delta_N \in Q_N$ is exactly paved by tesserae of Q_{N+1}. We consider over the σ algebra generated by the

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cylinders of $\Omega_N = \{-1, 1\}^{Q_N}$ the equilibrium measure P_N of the Ising model over Q_N at large temperature β^{-1} and zero external field.[2] Let us define the space $\Omega^{(N)} = \prod_{k=0}^{N} \Omega_k$ and the product measure $P^{(N)} = \prod_{k=0}^{N} P_k$. We then assume as free field with ultraviolet cutoff the family of random variables over $\Omega^{(N)}$ indexed by R^2 defined by

$$\varphi_\xi^{(N)} = \sum_{k=0}^{N} \sigma_{\Delta_k(\xi)}, \qquad \xi \in R^2$$

where $\Delta_k(\xi)$ is the tessera of Q_k containing ξ and $\sigma_{\Delta_k} \in \{-1, 1\}$ denotes the Ising spin variable at Δ_k, and also its natural extension to a random variable over $\Omega^{(N)}$.

There are some obvious remarks. The field $\varphi_\xi^{(N)}$, $\xi \in R^2$, is the sum of $N+1$ independent fields and

$$|\varphi_\xi^{(N)}| \leqslant N+1$$

$\varphi^{(N)}$ is constant in each tessera Δ_N; it has zero mean

$$\langle \varphi_\xi^{(N)} \rangle = 0$$

and covariance

$$\langle \varphi_\xi^{(N)} \varphi_\eta^{(N)} \rangle = \sum_{k=0}^{N} \langle \sigma_{\Delta_k(\xi)} \sigma_{\Delta_k(\eta)} \rangle$$

The covariance has a sort of logarithmic divergence as $|\xi - \eta| \to 0$ because if $\xi, \eta \in \Delta_{N-1}$ and $|\xi - \eta| > 2^{-N}$, it follows that

$$\langle \varphi_\xi^{(N)} \varphi_\eta^{(N)} \rangle > \log_2 |\xi - \eta|^{-1}$$

Since we want to study an interacting theory of φ^4 type, we must introduce an interaction containing subtractions depending on the measure $P^{(N)}$ we have chosen. This can be done using the general definition of Wick powers contained in Ref. 3, as summarized below.

If x is a random variable over (X, Σ, μ) with finite moments, the Wick powers of x are defined recursively by

$$:x^0: = 1$$

$$\frac{d}{dx} :x^n: = n :x^{n-1}: \qquad (n \geqslant 1)$$

$$\langle :x^n: \rangle = 0 \qquad (n \geqslant 1)$$

We obtain, for instance, if $\langle x^k \rangle = 0$ for k odd,

$$:x^4: = x^4 - 6\langle x^2 \rangle x^2 - \langle x^4 \rangle + 6\langle x^2 \rangle^2$$

So we have, for each $N \geqslant 0$, after an easy computation,

$$:\varphi_\xi^{(N)4}: = \varphi_\xi^{(N)4} - 6(N+1)\varphi_\xi^{(N)2} + 3N^2 + 8N + 5$$

We define the renormalized interaction as

$$V_\Lambda^{(N)} = \lambda \int_\Lambda d\xi : \varphi_\xi^{(N)4} :$$

where λ is a positive constant and Λ is a bounded regular region of R^2. $V_\Lambda^{(N)}$ enjoys the following property that will be central in removing the ultraviolet divergence

$$\int dP_N V_\Lambda^{(N)} = V_\Lambda^{(N-1)} \tag{1}$$

This follows from the equation that holds if x and y are independent:

$$:(x+y)^n: = \sum_{k=0}^{n} \binom{n}{k} :x^k: \, :y^{n-k}:$$

putting $\varphi_\xi^{(N)} = \varphi_\xi^{(N-1)} + \sigma_{\Delta_N(\xi)}$. Furthermore there is a constant $a > 0$ such that for $N \geqslant 0$

$$|V_\Lambda^{(N)}| \leqslant \lambda a N^4 |\Lambda|$$

where $|\Lambda|$ is the area of Λ.

The partition function is

$$Z_\Lambda^{(N)} = \int dP^{(N)} e^{-V_\Lambda^{(N)}}$$

and the pressure with ultraviolet and infrared cutoff is

$$p_\Lambda^{(N)} = \frac{1}{|\Lambda|} \log Z_\Lambda^{(N)}$$

We work in a range of temperature given by $\beta \leqslant \beta_0$, where β_0, defined below, is smaller than the inverse critical temperature. The following proposition is our result.

Proposition 1 (Existence of the Pressure). Let $\beta \leqslant \beta_0$; then the limit

$$\lim_{\substack{N\to\infty \\ \Lambda\to\infty}} p_\Lambda^{(N)}$$

exists if the limit in Λ is taken in the sense of Van Hove. (For the Van Hove limit see Ref. 4.) In the proof we will use some important properties of the measure P_N, well known in the theory of the two-dimensional Ising model. We expose them briefly.

Denote by Λ_N a finite subset of R^2 paved by Q_N, $\tilde{\Lambda}_N$ its complement, and $\partial\Lambda_N$ the external boundary of Λ_N thought of as a subset of Q_N. Let σ_{Λ_N} and $\sigma_{\tilde{\Lambda}_N}$ be configurations over Λ_N and $\tilde{\Lambda}_N$ and $P_N(d\sigma_{\Lambda_N} | \sigma_{\tilde{\Lambda}_N})$ the conditional probability of the cylindrical event defined by σ_{Λ_N} with respect to the σ algebra of the events with base in $\tilde{\Lambda}_N$. P_N has the Markov property, i.e.,

$$P_N(d\sigma_{\Lambda_N} | \sigma_{\tilde{\Lambda}_N}) = P_N(d\sigma_{\Lambda_N} | \sigma_{\partial\Lambda_N})$$

Let Λ'_N be a finite subset at a distance $d_N(\Lambda_N, \Lambda'_N)$ from Λ_N, where d_N is the Euclidean distance in units of 2^{-N}. We define $\eta(\sigma_{\Lambda_N}, \sigma_{\Lambda'_N})$ such that

$$P_N(d\sigma_{\Lambda_N} | \sigma_{\Lambda'_N}) = P_N(d\sigma_{\Lambda_N}) \exp \eta(\sigma_{\Lambda_N}, \sigma_{\Lambda'_N})$$

The following proposition holds.[5]

Proposition 2. There is a positive constant A and a function $\chi(\beta)$ such that $\lim_{\beta\to 0}\chi(\beta) = +\infty$ and

$$|\eta(\sigma_{\Lambda_N}, \sigma_{\Lambda'_N})| \leqslant \min\{[\partial\Lambda_N], [\partial\Lambda'_N]\} A \exp[-\chi(\beta) d_N(\Lambda_N, \Lambda'_N)]$$

($[\partial\Lambda_N]$ is the number of tesserae of $\partial\Lambda_N$.)

This proposition allows us to relate the integrals $\int dP_N V_{C_N}^{(N)}$ and $\int dP_N \exp(-V_{C_N}^{(N)})$ to the conditioned integrals $\int dP_N(B_N) V_{C_N}^{(N)}$ and $\int dP_N(B_N)\exp(-V_{C_N}^{(N)})$, where the sequences of sets $\{B_N\}_{N\geqslant 1}$ and $\{C_N\}_{N\geqslant 1}$ are such that: the sequences of their areas are bounded, the sets $\{\Delta_N | \Delta_N \cap B_N \neq \varnothing\}$ and $\{\Delta_N | \Delta_N \cap C_N \neq \varnothing\}$ are disjoint, $[\partial B_N] = 0(2^N)$, and $P_N(B_N)$ denotes the measure P_N conditioned to the variables over the set $\{\Delta_N | \Delta_N \cap B_N \neq \varnothing\}$.

In the case $d(B_N, C_N) > 1$, Proposition 2, with $d_N(B_N, C_N) > 2^N$ gives, introducing an obvious notation,

$$\int dP_N(B_N)\exp[-V_{C_N}^{(N)}]$$
$$\lessgtr \int dP_N \exp[-V_{C_N}^{(N)}] \exp\{\pm[\partial B_N] A \exp[-\chi(\beta)2^N]\}$$

Let be β_0 such that for $\beta \leqslant \beta_0$, $\chi(\beta) > 1$. In this range of temperature, which we assume from now on, the argument of the exponential is $[\partial B_N]$ $O(e^{-2^N})$. In the general case we introduce $\overline{C}_N = \{\Delta_N \cap C_N \neq \varnothing | d_N(\Delta_N, B_N) \leqslant N\}$, decompose the energy

$$V_{C_N}^{(N)} = V_{\overline{C}_N}^{(N)} + V_{C_N \setminus \overline{C}_N}^{(N)}$$

and write

$$\int dP_N(B_N) V_{C_N}^{(N)} = \int dP_N(B_N) V_{\overline{C}_N}^{(N)} + \int dP_N(B_N) V_{C_N\setminus\overline{C}_N}^{(N)}$$

We apply Proposition 2, with $d_N(B_N, C_N \setminus \overline{C}_N) > N$:

$$\left|\int dP_N(B_N) V_{C_N\setminus\overline{C}_N}^{(N)} - \int dP_N V_{C_N\setminus\overline{C}_N}^{(N)}\right| \leqslant \lambda a N^4 |C_N| |\exp(A[\partial B_N]e^{-N}) - 1| \tag{2}$$

Using Eq. (1) we can write

$$\int dP_N(B_N) V_{C_N}^{(N)} \lessgtr \int dP_N(B_N) V_{\overline{C}_N}^{(N)} + V_{C_N\setminus\overline{C}_N}^{(N-1)} \pm O(N^4 2^N e^{-N}) |C_N|$$

From

$$|V_{\bar{C}_N}^{(N)}| \leqslant \lambda a N^4 |\bar{C}_N|$$

we have

$$\int dP_N(B_N)\exp\left(-V_{C_N}^{(N)}\right) \lessgtr \int dP_N(B_N)\exp\left(-V_{C_N \setminus \bar{C}_N}^{(N)}\right)\exp\left(\pm\lambda a N^4|\bar{C}_N|\right)$$

We can conveniently estimate the error in the case that

$$|\{\Delta_N \,|\, d(\Delta_N, B_N) \leqslant N\}| = [\partial B_N] O(N 2^{-2N})$$

In fact we have, *a fortiori*, $|\bar{C}_N| = O(N2^{-N})$ and so the error is $O(N^5 2^{-N})$. Using Proposition 2 we find

$$\int dP_N(B_N)\exp\left(-V_{C_N \setminus \bar{C}_N}^{(N)}\right) \lessgtr \int dP_N \exp\left(-V_{C_N \setminus \bar{C}_N}^{(N)}\right)\exp\left(\pm A[\partial B_N] e^{-N}\right)$$

and finally we can write

$$\int dP_N(B_N)\exp\left(-V_{C_N}^{(N)}\right)$$
$$\lessgtr \int dP_N \exp\left(-V_{C_N}^{(N)}\right)\exp\left(\pm 2\lambda a N^4|\bar{C}_N| \pm A[\partial B_N]e^{-N}\right)$$

We summarize the above considerations, which are the basic ingredients in the proof of our results, in the following lemma.

Lemma 1. Let be $\beta \leqslant \beta_0$:

$$\int dP_N(B_N) V_{C_N}^{(N)} \lessgtr \int dP_N(B_N) V_{C_N}^{(N)} + V_{C_N \setminus \bar{C}_N}^{(N-1)} \pm O(N^4 2^N e^{-N})|C_N| \quad (3)$$

If $\{B_N\}_{N \geqslant 1}$ is such that $|\{\Delta_N \,|\, d_N(\Delta_N, B_N) \leqslant N\}| = [\partial B_N] O(N 2^{-2N})$, then

$$\int dP_N(B_N)\exp\left(-V_{C_N}^{(N)}\right)$$
$$\lessgtr \int dP_N \exp\left(-V_{C_N}^{(N)}\right)\exp\left\{\pm[\partial B_N] O(N^5 2^{-2N} + e^{-N})\right\} \quad (4)$$

If $d(B_N, C_N) > 1$, then

$$\int dP_N(B_N)\exp\left(-V_{C_N}^{(N)}\right) \lessgtr \int dP_N \exp\left(-V_{C_N}^{(N)}\right)\exp\left\{\pm[\partial B_N] O(e^{-2^N})\right\} \quad (5)$$

2. THE STABILITY

The following proposition holds.

Proposition 3 (Stability). There is a positive constant E such that for each $N \geqslant 0$ and Λ

$$e^{-E|\Lambda|} \leqslant Z_\Lambda^{(N)} \leqslant e^{E|\Lambda|}$$

We are going to derive this proposition as a corollary of Lemma 2 which, in turn, can be derived using only statistical mechanics arguments.

Lemma 2. There exists a summable sequence of positive numbers $\{E_N\}_{N\geqslant 1}$ such that for each $N \geqslant 1$ and Λ,

$$\exp\left(-V_\Lambda^{(N-1)} - E_N|\Lambda|\right) \leqslant \int dP_N \exp\left(-V_\Lambda^{(N)}\right) \leqslant \exp\left(-V_\Lambda^{(N-1)} + E_N|\Lambda|\right) \tag{6}$$

In fact we can write $Z_\Lambda^{(N)}$ in the form $\int dP_0 \cdots \int dP_N \exp(-V_\Lambda^{(N)})$ and apply Eq. (6) repeatedly for $N \geqslant 1$. We are led to the trivial case $V_\Lambda^{(0)} = 0$ and then the proposition follows putting $E = \sum_{N\geqslant l} E_N$.

Proof. We introduce the infinite square grid G_N, paved by Q_N, formed by strips of breadth 2^{-N} and having a spacing of $8N^7 2^{-N}$. We call $\square_N$ the squares individuated by G_N and write

$$\int dP_N \exp\left(-V_\Lambda^{(N)}\right) = \int dP_N \exp\left(-V_{G_N\cap\Lambda}^{(N)}\right)$$
$$\times \int dP_N(G_N \cap \Lambda) \prod_{\square_N \cap \Lambda \neq \varnothing} \exp\left(-V_{\square_N\cap\Lambda}^{(N)}\right) \tag{7}$$

The Markov property of P_N allows to evaluate the internal integral as a product of integrals in the conditioned measure $P_N(\partial\square_N)$:

$$\prod_{\square_N\cap\Lambda\neq\varnothing} \int dP_N(\partial\square_N) \exp\left(-V_{\square_N\cap\Lambda}^{(N)}\right)$$

We apply the second-order Taylor formula to the function of λ $\log \int dP_N(\partial\square_N) \exp(-V_{\square_N\cap\Lambda}^{(N)})$, with initial point $\lambda = 0$:

$$\int dP_N(\partial\square_N) \exp\left(-V_{\square_N\cap\Lambda}^{(N)}\right) = \exp\left[-\int dP_N(\partial\square_N) V_{\square_N\cap\Lambda}^{(N)} + \epsilon_{\square_N\cap\Lambda}\right]$$

The rest $\epsilon_{\square_N\cap\Lambda}$ is easily bounded:

$$|\epsilon_{\square_N\cap\Lambda}| \leqslant \left(\lambda a N^4 |\square_N \cap \Lambda|\right)^2 \exp(4\lambda a N^4 |\square_N \cap \Lambda|)$$
$$= O(N^{22} 2^{-2N}) |\square_N \cap \Lambda|$$

We apply Eq. (3) of Lemma 1 and get

$$\int dP_N(\partial\square_N) \exp\left(-V_{\square_N\cap\Lambda}^{(N)}\right) \lesseqgtr \exp\left[-\int dP_N(\partial\square_N) V_{\square_N\cap\Lambda}^{(N)}\right.$$
$$\left. - V_{(\square_N\setminus\bar{\square}_N)\cap\Lambda}^{(N-1)} \pm O(N^{22} 2^{-2N} + N^4 2^N e^{-N}) |\square_N \cap \Lambda|\right] \tag{8}$$

Using the Taylor formula in the reverse direction we find

$$\exp\left[-\int dP_N(\partial\square_N)\, V^{(N)}_{\bar{\square}_N\cap\Lambda}\right] \leqslant \int dP_N(\partial\square_N)\exp\left(-V^{(N)}_{\bar{\square}_N\cap\Lambda}\right)\exp\left[\pm O(N^{22}2^{-2N})|\square_N\cap\Lambda|\right]$$

and so the bound (8) becomes

$$\int dP_N(\partial\square_N)\exp\left(-V^{(N)}_{\square_N\cap\Lambda}\right)\lessgtr\int dP_N(\partial\square_N)\exp\left(-V^{(N)}_{\bar{\square}_N\cap\Lambda}\right) \times\exp\left(-V^{(N-1)}_{(\square_N\setminus\bar{\square}_N)\cap\Lambda}\right)\exp\left[\pm O(N^{22}2^{-2N}+N^4 2^N e^{-N})|\square_N\cap\Lambda|\right]$$

Equation (7) and the above bound give

$$\int dP_N\exp(-V^{(N)}_\Lambda) \lessgtr\exp\left(-V^{(N-1)}_{\Lambda^0_N}\right)\int dP_N\exp\left(-V^{(N)}_{\Lambda\setminus\Lambda^0_N}\right)\exp\left[\pm(1/3)E_N|\Lambda|\right] \quad (9)$$

where we have put $\Lambda^0_N=\cup_{\square_N\cap\Lambda\neq\varnothing}(\square_N\setminus\bar{\square}_N)\cap\Lambda$ and $E_N=3O(N^{22}2^{-2N}+N^4 2^N e^{-N})$. We are so led to evaluate $\int dP_N\exp(-V^{(N)}_{\Lambda\setminus\Lambda^0_N})$. We apply the bound just obtained introducing a new grid G^1_N, with the same spacing of G_N, whose vertices are at the centers of the squares $\square_N$. In place of Λ^0_N we will find Λ^1_N and so we are led to evaluate $\int dP_N\exp(-V^{(N)}_{\Lambda\setminus\Lambda^0_N\setminus\Lambda^1_N})$. A further application of Eq. (9) with an obvious choice of the grid G^2_N reduces the estimate to the trivial case $\Lambda\setminus\Lambda^0_N\setminus\Lambda^1_N\setminus\Lambda^2_N=\varnothing$ and gives Eq. (6). ■

In the proof of the lemma we do not make essential use of the positivity of λ; we could consider the general case introducing only obvious modifications.

The technique used in the proof can be suitably extended to show the asymptotic convergence of the formal Taylor series in λ of $p^{(N)}_\Lambda$, that we write

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\epsilon^T\left(V^{(N)}_\Lambda;k\right)$$

In other words we have to show that for each t

$$\exp\left[|\Lambda|\sum_{k=0}^{t}\frac{\lambda^k}{k!}\epsilon^T\left(V^{(N)}_\Lambda;k\right)-E_t(\lambda)|\Lambda|\right]\leqslant Z^{(N)}_\Lambda \leqslant\exp\left[|\Lambda|\sum_{k=0}^{t}\frac{\lambda^k}{k!}\epsilon^T\left(V^{(N)}_\Lambda;k\right)+E_t(\lambda)|\Lambda|\right] \quad (10)$$

where $E_t(\lambda)=o(\lambda^t)$.

For $t = 1$ the above assertion is

$$\exp[-E_1(\lambda)|\Lambda|] \leqslant Z_\Lambda^{(N)} \leqslant \exp[E_1(\lambda)|\Lambda|]$$

with $E_1(\lambda) = o(\lambda)$. To get this result we have only to show, by Proposition 3, that $E = o(\lambda)$. We observe, referring to the proof of Lemma 2, and in particular to Eq. (9), that E_N is the sum of two contributions: the first deriving from $\epsilon_{\square_N \cap \Lambda}$, the second from Eq. (3). The first is $O(\lambda^2)$ because we have used the second-order Taylor formula; the second is $O(\lambda)$ but it can be done smaller easily. In fact we define, fixed t,

$$\overline{C}_N(\lambda) = \{\Delta_N \,|\, \Delta_N \cap C_N \neq \emptyset, \quad d_N(\Delta_N, B_N) \leqslant Nt \log(e + 1/\lambda)\};$$

so the coefficient of $|C_N|$ in Eq. (2) is $O(\lambda\lambda^{tN})$ and gives a contribution $O(\lambda^{t+1})$ to E. To guarantee the applicability of Eq. (9) we must choose for the grid G_N a spacing $8N^7 2^{-N} t \log(e + 1/\lambda)$. This implies that the contribution of $\epsilon_{\square_N \cap \Lambda}$ to E is now $O(\lambda^2 \log^2(e + 1/\lambda))$. In the general case Eq. (10) can be shown in the following way: we need introduce the interaction

$$V_\Lambda^{(N,t)} = V_\Lambda^{(N)} + |\Lambda| \sum_{k=0}^{t} \frac{\lambda^k}{k!} \epsilon^T(V_\Lambda^{(N)}; k)$$

and prove that the related partition function $Z_\Lambda^{(N,t)}$ satisfies

$$\exp[-E_t(\lambda)|\Lambda|] \leqslant Z_\Lambda^{(N,t)} \leqslant \exp[E_t(\lambda)|\Lambda|]$$

which is precisely Eq. (10). For the proof we refer to the second reference in Ref. 1 in which explicit computations are made for the Gaussian case.

3. THE PRESSURE

We prove Proposition 1 by the two following points:

(1) For each Λ, $p_\Lambda^{(N)}$ has a limit when N tends to infinity;

(2) The sequence $p_\Lambda^{(N)}$ has a limit when Λ tends to infinity (Van Hove) and the limit is reached uniformly in N.

A first step in the proof of point (1) can be easily done: it is the monotonicity in N of $p_\Lambda^{(N)}$. We write

$$p_\Lambda^{(N)} = \frac{1}{|\Lambda|} \log \int dP^{(N-1)} \int dP_N \exp(-V_\Lambda^{(N)})$$

and apply the Jensen inequality and Eq. (1):

$$\int dP_N \exp(-V_\Lambda^{(N)}) \geqslant \exp\left(-\int dP_N V_\Lambda^{(N)}\right) = \exp(-V_\Lambda^{(N-1)})$$

and so

$$p_\Lambda^{(N)} \geqslant p_\Lambda^{(N-1)}$$

The stability implies the boundedness of the sequence $p_\Lambda^{(N)}$, and so point (1) follows.

We solve the problem in point (2) using the usual strategy for the proof of the existence of the thermodynamic limit[6]: we first show in Lemma 3 that the limit in Λ of $p_\Lambda^{(N)}$ exists and is reached uniformly in N for a particular sequence of squares, then extend this result to any Van Hove sequence. Such an extension is in our case obvious and we do not expose it.

Lemma 3. Let $\{\Lambda_s\}_{s\geqslant 1}$ be the sequence of the squares, paved by Q_0, with sides $a_s = 4(2^s - 1)$. Then the limit

$$\lim_{s\to\infty} p_{\Lambda_s}^{(N)}$$

exists and is uniform in N.

Proof. We observe that from $a_{s+1} = 2a_s + 4$, Λ_{s+1} can be divided into four squares Λ_s^i, $i = 1, \ldots, 4$, having distance 1 from the boundary of Λ_{s+1} and distance 2 between them. Our aim is to show that for each N

$$Z_{\Lambda_s}^{(N)4}\exp\left[-H(|\Lambda_{s+1}|\backslash 4|\Lambda_s|)\right] \leqslant Z_{\Lambda_{s+1}}^{(N)} \leqslant Z_{\Lambda_s}^{(N)4}\exp\left[H(|\Lambda_{s+1}|\backslash 4|\Lambda_s|)\right] \tag{11}$$

where H is a positive constant (independent of N and s). In fact from this equation we get

$$|P_{\Lambda_{s+1}}^{(N)} - p_{\Lambda_s}^{(N)}| \leqslant |p_{\Lambda_s}^{(N)}|\left(1 - \frac{4|\Lambda_s|}{|\Lambda_{s+1}|}\right) + H\frac{|\Lambda_{s+1}| - 4|\Lambda_s|}{|\Lambda_{s+1}|}$$

By the stability $|p_{\Lambda_s}^{(N)}| \leqslant E$ and so the right-hand member is bounded by a sequence in s, independent of N, that for the particular choice of $\{\Lambda_s\}_{s\geqslant 1}$ is summable. This implies that $\{P_{\Lambda_s}^{(N)}\}_{s\geqslant 1}$ is a Cauchy sequence and reaches its limit uniformly in N.

In order to show Eq. (11) we put $B = \cup_{i=1}^4 \Lambda_s^i$, $C = \Lambda_{s+1}\backslash B$ and $P^{(N)}(B) = \prod_{k=0}^N P_k(B_k)$, where $B_k = \{\Delta_k \,|\, \Delta_k \subset B\}$. We introduce a conditioning over B:

$$Z_{\Lambda_{s+1}}^{(N)} = \int dP^{(N)}\exp(-V_B^{(N)})\int dP^{(N)}(B)\exp(-V_C^{(N)}) \tag{12}$$

and prove that there is a positive constant D such that

$$\int dP^{(N)}(B)\exp(-V_C^{(N)}) \lesseqgtr \exp(\pm D|\partial B| \pm E|C|) \tag{13}$$

where $|\partial B|$ is the length of the boundary of B. In fact, by Lemma 1, Eq. (4), we have

$$\int dP_N(B_N)\exp(-V_C^{(N)}) \lesseqgtr \int dP_N$$
$$\times\exp(-V_C^{(N)})\exp\left\{\pm\left[\partial B_N\right]O(N^5 2^{-2N} + e^{-N})\right\}$$

From that and Lemma 2

$$\int dP_N(B_N)\exp(-V_C^{(N)}) \lessgtr \exp(-V_C^{(N-1)})$$
$$\times \exp\{\pm E_N|C| \pm [\partial B_N] O(N^5 2^{-2N} + e^{-N})\}$$

Iterating on N, and using $[\partial B_N] = |\partial B|2^N$, Eq. (13) follows. Equation (12) becomes

$$Z_{\Lambda_{s+1}}^{(N)} \lessgtr \int dP^{(N)} \exp(-V_{\cup_i \Lambda_s^i}^{(N)}) \exp[\pm(E+D)(|\Lambda_{s+1}| - 4|\Lambda_s|)]$$

It remains to show

$$\int dP^{(N)} \exp(-V_{\cup_i \Lambda_s^i}^{(N)}) \lessgtr Z_{\Lambda_s}^{(N)4} \exp[\pm 4F|\partial\Lambda_s|]$$

where F is a positive constant. We introduce a conditioning over Λ_s^1 and write

$$\int dP^{(N)} \exp(-V_{\cup_i \Lambda_s^i}^{(N)}) = \int dP^{(N)} \exp(-V_{\Lambda_s^1}^{(N)}) \int dP^{(N)}(\Lambda_s^1) \exp(-V_{\cup_{i\neq 1}\Lambda_s^i}^{(N)})$$

Proceeding as before we are led to apply Eq. (5) of Lemma 1, and iterating on N we get

$$\int dP^{(N)}(\Lambda_s^1) \exp(-V_{\cup_{i\neq 1}\Lambda_s^i}^{(N)}) \lessgtr \int dP^{(N)} \exp(-V_{\cup_{i\neq 1}\Lambda_s^i}^{(N)}) \exp(\pm F|\partial\Lambda_s^1|)$$

From that the lemma follows.

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